

Coordinate potential approach to space-time curvature

Robert L. Shuler¹

¹NASA Johnson Space Center/EV5, 2101 NASA Parkway, Houston, TX 77058
robert.l.shuler@nasa.gov or robert@mc1soft.com

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Abstract: A direct approach to curvature might provide physical, pedagogical and verification insight even if it does not fall within classic weak-field methods. Differential uniform field space-time elements are formulated in a space capable of curvature, i.e. not a parallel field, without using free fall, so there is no ambiguity of kinetic vs. static effects. Working backward from known metrics we determine it is necessary to postulate an area-coordinate formulation of potential energy, for which we present additional rationale. A specific field strength postulate produces an exact metric, and the necessary postulate for the Schwarzschild metric is identified. Plausible field postulates are given for other metrics as well, to quantify verification issues raised by prior investigators.

Key words: gravity; general relativity; equivalence; space-time; curvature; time dilation; length contraction

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1. Introduction

In this paper we explore physical arguments that support space-time curvature, with the Schwarzschild metric emerging only after an additional specification of a particular field characteristic. While a “simple” derivation of the static spherical gravitational field metric using the field equation is available [1], it does not explain or resolve interesting questions that have arisen from many attempts to derive metrics directly, without the mathematical overhead of tensors and partial differential equations. Usually these methods start from Einstein’s equivalence principle (EEP) and proceed with drop experiments. [2] [3] Generally they are considered to have failed [4] [5], and have led to suggestions that such methods are not possible [6], or at least that spatial curvature is not logically necessary within them. [7] But only full light bending has been conclusively shown to be not evident in an accelerated equivalence setup. [8] [9] And conversely, one finds opinions such as that of Clifford Will from his book on the verification of gravity: “...it is possible to argue convincingly that if the EEP is valid, then gravitation must be a curved space-time phenomenon,” [10] implying sufficiency of the EEP.

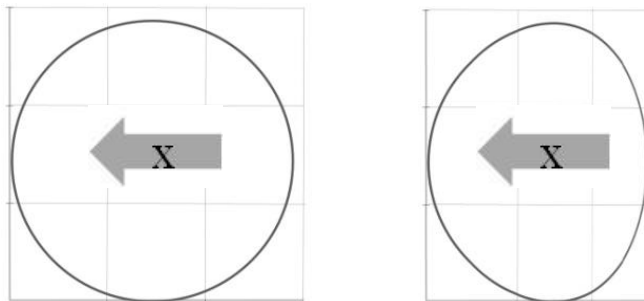


Fig. 1 – Euclidean circle (L) and progressively contracted in x (R), both “flat”

However, arguing from the EEP involves first determining a uniform parallel field metric, and it is well known that curvature is not useful for describing such metrics. Even the limited goal of finding length changes is plagued by lack of a reference, illustrated in Figure 1. Any coordinate length moved to the right contracts, and cannot be used to detect a change in objects. Both circles are flat and $C=2\pi r$. This is not a deficiency in the EEP. It is a *statement* of the EEP. Free fall observers in a uniform parallel field (Einstein's sufficiently small laboratory) cannot distinguish their environment from an inertial reference frame free of gravitational effects. Nor can accelerated observers distinguish the two. In part this is due to lack of any point in the space with a different acceleration, or a lack of gravity, with which to compare. The EEP provides heuristic guidance that gravity may be represented as geometry, not a prescription to find it.

Therefore we cannot begin in a parallel field. We must use a field topology that allows curvature to be discovered. The simplest choice is probably spherical symmetry, which has an area-coordinate, usually designated "r" but defined in terms of area (or circumference) in a way that is common to distant and remote observers. The hypothetical radius "r" is a real physical radius only in flat space-time, i.e. without any attractor mass at $r=0$.

However, this alone is not sufficient either. Using only Special Relativity (SR) we will not be able to find curvature. Consider the Schwarzschild metric, which defines time dilation as a function of the area-coordinate "r," and also gives a relation between proper and coordinate lengths. There is not exactly a notion of proper and coordinate lengths in SR. Time dilation and length contraction are mutually perceived between any pair of observers and neither has priority. We should be able to see gravitational effects without relative motion, and if relative motion is present (i.e. if we use a "drop experiment") then there may be some ambiguity as to whether we are looking at motion effects or gravitational effects.

While Gruber has given a mathematical argument, we add a conceptual one as follows: There are multiple theories of gravity which obey the EEP, have spherically symmetric solutions, and obey SR. The Schwarzschild metric is a unique solution in one of them, General Relativity (GR). Assuming at least two of those theories are self-consistent, and the Schwarzschild solution appears in only one of them, then it should not be uniquely implied by the commonly held postulates. An additional postulate is logically necessary. We will work backward from the Schwarzschild solution (because the calculus is easier in that direction) to find what that postulate should be. This postulate will then be a result our work, a subject for discussion, not merely an assumption for its beginning.

2. The form of a spherically symmetric metric

We suppose "metric gravity" without detailed discussion. UFF holds that all objects, matter or energy, experience identical influences to motion at each point. Then geometry, which affects all objects equally, can be used to describe the influences to motion. Several authors [e.g. Ibid. 6] have pointed out that using static spherically symmetric gravity and the Schwarzschild coordinates (t, r, θ, ϕ) , cross terms disappear, and the metric relating the coordinates to a local Riemannian space-time interval (ds , or $d\tau$) will have the form:

$$ds^2 = g_{tt}c^2 dt^2 + g_{rr}dr^2 + r^2(d\theta^2 + \sin\theta d\phi^2) \quad (1)$$

This assumes that lengths which are transverse to the radius are not physically changed, which is usual, and critical to our method. The g_{tt} and g_{rr} are functions of r , for which a typical usage is [Ibid. 6]:

$$-F(r)^2 = g_{tt}, H(r)^2 = g_{rr}$$

If the time coefficient $F(r)$ is known, it is only necessary to find one independent constraint to solve for the spatial coefficient $H(r)$. Since we will use $F(r)$ for force, it will be convenient to make the substitutions:

$$\Upsilon_T = F(r), \Upsilon_R = H(r)$$

For time relations at a fixed location $ds = 0$, thus $\Upsilon_T = d\tau / dt$. In a static field at a fixed location $d\theta = d\phi = dt = 0$, and we have $\Upsilon_R = ds / dr$. From the Schwarzschild solution we know that:

$$\Upsilon_R = 1 / \Upsilon_T \quad (2)$$

3. Working backward using the force on a tether

We use an ideal tether that neither absorbs nor gives up energy, i.e., it undergoes neither elastic nor inelastic deformation, and has no mass of its own. However, it does undergo relativistic length contraction. Thus if it moves a proper length dl at one end, it moves dl at the other also. Such thought experiment devices were used by Einstein, suggested to the author from a book by Leonard Susskind [11], and find theoretical application even over cosmic distances in the tethered galaxy problem. [12] The tether is not entirely necessary but has two functions. It is a device for visualizing coordinate transformations, making them less abstract. More importantly, it is a device for conducting thought experiments in which relative velocities are held to trivial levels so that SR effects are avoided, for reasons mentioned above.

Following a derivation given by Kassner of the force on a tether for an infinitely distant observer in Schwarzschild coordinates [13], if a mass m is lowered in the field, its energy decreases due to time dilation:

$$E(r) = \Upsilon_T mc^2 = (1 - 2GM / rc^2)^{1/2} mc^2 \quad (3)$$

The decrease in energy is extracted on the tether by a force F , acting through a length dl of tether spooled:

$$Fdl = -dE \quad (4)$$

From the known metric we have:

$$dl = -\Upsilon_R dr = -(1 - 2GM / rc^2)^{-1/2} dr \quad (5)$$

Using (3) we find:

$$dE = dr \frac{d}{dr} E(r) = dr (mc^2) (1 - 2GM / rc^2)^{-1/2} GM / r^2 c^2 \quad (6)$$

Inserting dE and dl into (4), the inverse square root terms cancel:

$$F = -dE / dl = [dr (mc^2) GM / r^2 c^2] / dr = GMm / r^2 \quad (7)$$

Thus in the Schwarzschild coordinates, for a distant observer, the static spherically symmetric solution to GR reduces to Newton's gravity. The point of most interest to us, however, is that we have a relation between F , dl and dr . Under quasi-static conditions, for radial displacement, dl is a proxy for ds . Thus we have learned, by working backward from the answer, that if we already have the time coefficient, and we can pin down F by independent means, then we can find ds/dr and the spatial coefficient.

We can find force at some non-infinite radius, r_l . Such an observer, owing to the time dilation factor, experiences all energies increased by $(1 - 2GM / rc^2)^{-1/2}$. The proper length dl is invariant, so we have:

$$F_{r_l} = (1 - 2GM / rc^2)^{-1/2} GMm / r^2 = F_\infty / \Upsilon_T \quad (8)$$

4. Coordinate reference in a uniform field

Now we “forget” that we know (2) and begin to look for constraints on the coefficients. First we will find the time coefficient, i.e. an energy relation similar to (3), but generic for a small differential in a uniform field. Easy ways to find the time coefficient known: by Doppler, by a Special Relativity (SR) analysis, or using the Planck relation. The latter has the advantage that it works directly in a gravitational field, and does not require relative motion (which SR and Doppler both require).

What we need for Υ_r is a function of coordinate distance dr , not of the proper distance dl . Let A and B be coordinate stationary observers in a uniform field of strength a , with A higher than B. The proper distance between them, such as on an ideal tether, is dl . Ordinarily, using SR and the EPP, we’d assume a parallel field and a coordinate distance might, for example, be provided by inertial observers A_i and B_i momentarily co-moving with A and B. Their separation we’d designate dr . The problem with this approach, besides it being a “drop experiment,” is that we do not have a guarantee that the co-moving instant seen by A_i and B_i will also be simultaneous for A and B, whose clocks don’t even run at the same rate.

We note that proper distance can be meaningless in certain situations, for example it can change without potential changing. In Figure 2 (L) and (R) the circumference of a circle centered about an attractor mass M, of uniform density, is C in both cases, indicating a constant coordinate distance from A to the center of M. In 2 (L) the mass has a low density, let us suppose something like the sun as a gas giant, and so the proper distance from A to the center of M is very near to the coordinate distance. In 2 (R) suppose the sun, without a change in mass, has almost collapsed to a black hole. The excess proper radius is up to 7000 meters, but the potential of B is unchanged.

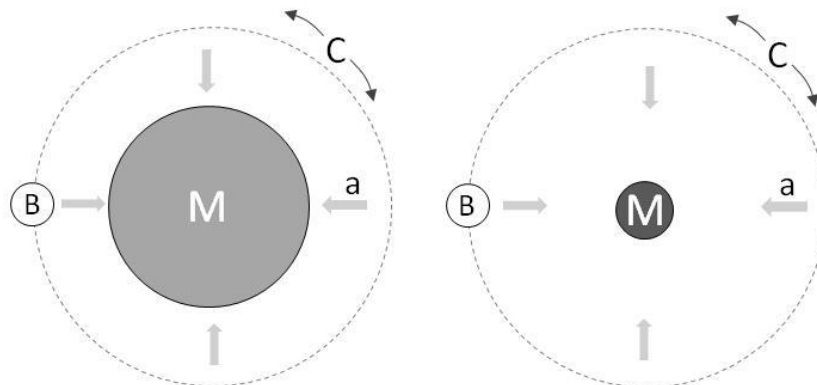
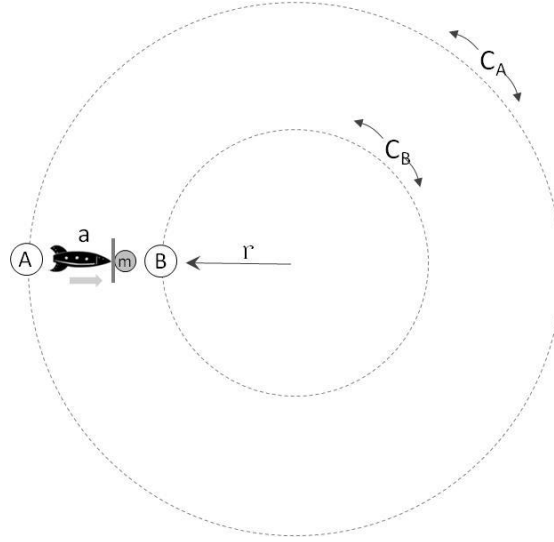


Fig. 2 – Potential of B is not dependent on proper radius changes.

Using this technique, the coordinate radius is visible to all observers by means of the reference circles. We construct an experiment with A and B located in the coordinate reference, but do not yet add the mass M. Instead, a rocket moves a small mass m between A and B as shown in Figure 3. The rocket need only apply a constant force $F = ma$, where a is the desired acceleration. It need not actually accelerate, for example, the small mass could be restrained by a tether so that the experiment proceeds slowly without necessity of accounting for relativistic velocities. In fact we require this approach, to avoid relativistic motion.


 Fig. 3 – Potential as a function of r and a

Without a gravitational field we have no reason to be suspicious of the length of the tether, and it reflects coordinate distance dr , in agreement with the reference circles. The acceleration is a function of r , i.e. $a(r) = d^2r / dt^2 = \text{constant}$. Taking a small difference in radius dr between A and B (the figure is exaggerated for illustration), the potential energy difference between them for m is, at low velocities:

$$dE_{AB} = -(ma)dr \quad (9)$$

Thus we have $E_B = E_A + dE_{AB} = E_A - (ma)dr$. We *postulate* that the time dilation factor is accurately predicted by this convention, a postulate that turns out to be true, both in a gravitational field and in accelerated frames in Minkowski space. In a gravitational field having coordinate acceleration a , we assert that (9) is valid for computing dE_{AB} . We call this the coordinate potential postulate. The use of the reference circles, rather than a falling inertial frame, allows measurement to be made quasi-statically so that A and B can be sure the interval corresponds to a measurement they will make by a different method.

5. Time coefficient in uniform field

To a scale factor of the number of photons produced by total annihilation of m , the locally measured energy is convertible to a frequency using the Planck relation:

$$E = mc^2 = hf \Rightarrow f = mc^2 / h \quad (10)$$

If we annihilate m at the bottom (B), the recovered frequency f_A at the top (A) must be red shifted exactly enough to account for the energy difference determined with the help of the coordinate reference, and thus we have time dilation as a function of coordinate distance. We use $d\tau$ for a time interval at the top, and dt for the corresponding proper time interval measured at B, and assume clocks which measure $d\tau$ and dt will run at speeds proportional to the Planck frequencies. The ratio $d\tau/dt$ necessary to obtain Y_T is found as the corresponding frequency ratio:

$$\begin{aligned} Y_T = d\tau / dt &= f_B / f_A = hf_B / hf_A = E_B / E_A = (E_A - (ma)dr) / E_A \\ \Leftrightarrow Y_T &= 1 - (ma)dr / E_A = 1 - (ma)dr / mc^2 = 1 - a(dr) / c^2 = 1 - d\Phi / c^2 \end{aligned} \quad (11)$$

6. The spatial coefficient

From (4), and using $dl = \Upsilon_R dr$, we infer a relation involving F , dE_A , dl , dr , and Υ_R :

$$dE_A = -F_A dl \Leftrightarrow F_A = -dE_A / dl = -(dE_A / dr) / \Upsilon_R \quad (12)$$

To pair with the proper distance dl , we use a proper force in the frame of A at the top of an ideal tether supporting m . For m local to B, the force is $F = ma$. Near A, the time at B appears dilated. What is the force at A? We have a transformation from GR (8), and using it we *would* have:

$$F_A = \Upsilon_T F_B \quad (13)$$

It is possible to give a justification independent of GR, several in fact, thus avoiding a circular argument: Assuming momentum conservation between A and B, at A we have a force $F_A = d\rho / dt$, but it is perceived at B as $F_B = d\rho / d\tau = (d\rho / dt)(dt / d\tau) = F_A / \Upsilon_T$ which reduces to (13).

The force at A is then $F_A = \Upsilon_T F_B = \Upsilon_T ma$. Substituting this into (12) we have:

$$F_A dl = -dE_{AB} \Leftrightarrow F_B \Upsilon_T dl = -dE \quad (14)$$

Recall that the force at B is, using gravitational sign:

$$F_B = -ma \quad (15)$$

Using (14) and (9) gives:

$$\begin{aligned} F_B \Upsilon_T dl &= -dE_{AB} = -ma(dr) \Leftrightarrow F_B = -ma(dr / dl) / \Upsilon_T \\ &\Leftrightarrow F_B = -ma / \Upsilon_R \Upsilon_T \end{aligned} \quad (16)$$

In view of (15) and (16), using $\Upsilon = \Upsilon_R$ to eliminate subscripts, we have therefore:

$$\Upsilon_R \Upsilon_T = 1 \Leftrightarrow \Upsilon_R = 1 / \Upsilon_T = \Upsilon \quad (17)$$

Substituting (17) into (1) gives a metric for a spherically symmetric uniform field:

$$ds^2 = -\Upsilon^{-2} c^2 dt^2 + \Upsilon^2 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (18)$$

We interpret (18) still only over differential distances, as a theoretical entity. In the next section we'll postulate physically reasonable field functions.

7. Metrics for non-uniform spherically symmetric static fields

The weak-field approximation

We can argue for the extension of (17) to non-uniform fields, if they are continuous and differentiable, by using stepwise uniform fields to approximate them, in the limit as steps become small. We now turn our attention to the question of what is required to obtain exactly the Schwarzschild coefficients. Most direct derivations will at this point use Newtonian potential from infinity Φ , with G the gravitational constant and M the gravitating mass, giving by approximation:

$$\Upsilon_R = 1 / \Upsilon_T = \Upsilon = (1 - \Phi / c^2)^{-1} = (1 - GM / rc^2)^{-1} \approx (1 - 2GM / rc^2)^{-1/2} \quad (19)$$

To obtain a “weak-field” approximation, we might use $\Upsilon \approx (1 - GM / rc^2 + \text{Order}(M^2 / r^2))^{-1}$, obtaining both time and spatial coefficients.

The Schwarzschild metric

We know from (8) that a distant observer would see gravity as the Newtonian inverse square force law. We can turn that around and reason the other way. If the Newtonian force law is postulated for a distant observer, the exact Schwarzschild coefficient is uniquely determined $\Upsilon = (1 - 2GM / rc^2)^{-1/2}$. The potential in this case would *not* be Newtonian.

Postulating the force law can be justified in the usual geometrical way, as the changing flux of “lines of force” through surfaces whose area varies as the square of coordinate radius, adding the idea that force is a $d\rho/dt$ process over each “line” which remains constant with respect to a fixed observer. Then it is unsurprising that GR was the first and is still the benchmark standard metric theory of gravity. It is relativistic Newtonian gravity as Einstein wished.

Field postulates corresponding to non-GR metric gravities

The question that motivated Schiff to suggest a direct derivation initially was verification. Our derivation allows for GR to be one of several metric theories of gravity, differing by various postulates, such as how to treat gravitational self-energy. If one took the Newtonian potential, for example, instead of the force law, one would obtain observations at 2 million miles from the sun differing in the 13th decimal place, presently indistinguishable in the solar system as atomic clocks would not survive in that thermal environment, but future observations should be able to detect such small differences.

For $\Upsilon = (1 - GM / rc^2)^{-1}$ the event horizon would be at a different place. $\Upsilon = (1 + \Phi / c^2)^{-1}$, which emerges from Schiff’s method (c.f. his (7)), has been used by Einstein in 1912 [14], Sciama in 1953 [15], and the author as an approximation for quantitatively illustrating Machian inertia. [16] If taken literally, there would be no event horizon, but rather an unforced singularity. For $\Phi / c^2 \gg 1$ the radial length of in-falling objects would collapse as fast as they approach $r=0$, creating an almost-black dot.

Many formulations are compatible with the EEP, as long as (17) holds. But they must also approach the Newtonian limit, requiring an approximation when $r \gg GM / c^2$, like (19), to be valid.

Alternative approaches and interpretations

Now that we see how the derivation works, we could derive the Schwarzschild field *only* using one fewer postulates. The coordinate potential postulate follows from the flux-area justification of the force law postulate as long as the area is normal to the acceleration vector, and our assumption about transverse lengths holds. The mechanics of the derivation can be exactly the same, except that having started with an expanded force law postulate, using it to obtain dependence on coordinate radii via the areas, we can only get the Schwarzschild coefficients. This may be preferred by some readers since coordinate radius only appears as a parameter of the surface areas, not as an artifact of a non-local coordinate system. The mathematics is identical. The philosophical implications are different. To obtain the non-GR metric theories, a dependence of potential or acceleration on the surface areas instead of coordinate radius is required, which is less natural. The author has given a brief sketch of a justification for a $1/r$ potential function using quantum superposition of three orthogonal $1/r$ position-momentum conjugate pair state

functions, though at that previous time a method for turning such a concept into curved space-time was not yet available. [17]

8. Conclusions

After clarifying that the role of the EPP is to justify a geometrical approach, and that it in fact forbids one to “find” gravity in uniform parallel fields, we looked to a postulate approach using non-parallel fields for a direct derivation of static gravity. Instead of a single postulate that produces a single exact metric theory, we used a two part approach in which a coordinate potential postulate defines a class of metric theories and provides insight about them.

Then specific field postulates or “laws of gravity” are used to pin down the static metrics of particular theories. An inverse square force law for coordinate stationary observers, matched with the Newtonian limit at infinity, leads to the Schwarzschild solution to GR for spherical static mass distributions. Comparisons to other plausible postulates, such as a potential law rather than a force law, provide the sort of heuristic guidance for verification that Schiff was looking for.

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